Eigenvalues and Eigenvectors

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Definition

Given a quadratic matrix $A \in \mathbb{R}^{n \times n}$ we call a vector $0 \neq v \in \mathbb{R}^n$ **eigenvector** if

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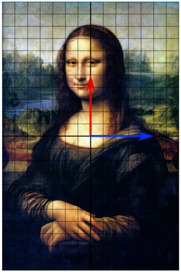
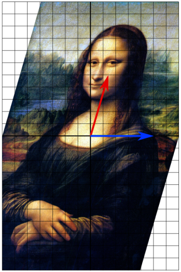
- An eigenvector is a vector whose direction is not changed under A





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- The (complex) roots of $P_A(\lambda)$ are the **eigenvalues** of A
- The **eigenvectors** are the solutions v of $(A - \lambda I_n)v = 0$

1. Compute $P_A(\lambda) = \det(A - \lambda I_n)$
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Notes:

- $P_A(\lambda) = 0$ has always n (possibly complex) solutions
- $\mu_A(\lambda)$ denotes the multiplicity of the root λ in P_A and is called **algebraic multiplicity**
- $\gamma_A(\lambda)$ denotes the number of *linear independent* eigenvectors for the eigenvalue λ and is called **geometric multiplicity**
- $1 \leq \gamma_A(\lambda) \leq \mu_A(\lambda) \leq n$

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- Suppose there are n linear independent eigenvectors $v_1, \ldots, v_n$. This is the case if and only if
  - $\mu_A(\lambda_i) = \gamma_A(\lambda_i)$ for $i = 1, \ldots, n$
  - In particular if $\mu_A(\lambda_i) = 1$ for all $i = 1, \ldots, n$ (i.e. all eigenvalues are different)
- Denote by $V := \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ the matrix of all eigenvectors and by $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ the diagonal matrix of all eigenvalues
- Then

$$A = VDV^{-1}$$